Cordial Labeling of Cartesian Product Between $K_{n,n} \times P_r$ and $K_{n,n} \times C_r$

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Abstract

Cordial labelling of a graph $G(V,E)$ with edge set $E$ and vertex set $V$ where the vertex set is label by $f : V \rightarrow \{0,1\}$ and each and every edge $xy$ is assigned the value $|f(x) - f(y)|$, where $f$ is known as cordial labelling if the number of vertices labelled by 0 and 1 which is differ by 1, and the number of edges labelled by 0 and 1 which is differ by 1 make this graph cordial is discussed here. If $|vf(0) - vf(1)| \leq 1$ and $|ef(0) - ef(1)| \leq 1$ are one of the main consideration. In this paper we pay our interest to label Cartesian product between complete bipartite graph $K_{n,n}$ and $P_r, K_{n,n}$ where $n$ is even and $C_r$. We also propose one algorithm to label the Cartesian product between $K_{n,n}$ and $P_r, K_{n,n}$ and $C_r$ where $n$ is even.

Keywords: Cordial Labelling, Graph Labelling and Cartesian Product of Graphs

I. INTRODUCTION

Graph labelling has many different applications in real world. Gallian [1] has published an effective survey on entire graph labelling and their different applications. The general graph labelling problems have a characteristics, where a set of numbers labels are preferred under a restricted environment, a rule may be imposed to each edge with a condition must satisfied as proposed by Hegde [2], Beineke and Hegde [3] proposed that graph labelling act as a border line between structure of graphs and number theory. This technique by which a graph is labelled can be applied on coding theory, missile guidance code. Labelling techniques successfully apply on X-ray crystallography and communication network also.

Cahit [4-14] has given the idea of both graceful [16] and harmonious [15] labelling. Cordial labelling of a graph $G(V,E)$ with edge set $E$ and vertex set $V$ where the vertex set is label by $f : V \rightarrow \{0,1\}$ and each and every edge $xy$ is assigned the value $|f(x) - f(y)|$, where $f$ is known as cordial labelling if the number of vertices labelled by 0 and 1 which is differ by 1, and the number of edges labelled by 0 and 1 which is differ by 1 make this graph cordial is discussed here. If $|vf(0) - vf(1)| \leq 1$ and $|ef(0) - ef(1)| \leq 1$ are one of the main consideration. In this paper we pay our interest to label Cartesian product between complete bipartite graph $K_{n,n}$ and $P_r, K_{n,n}$ where $n$ is even and $C_r$. We also propose one algorithm to label the Cartesian product between $K_{n,n}$ and $P_r, K_{n,n}$ and $C_r$ where $n$ is even.

Definition 1.1: Cartesian product of two graphs $G = (V,E)$ and $H = (V',E')$ is the Cartesian product between two set of vertices $V(G) \times V(H)$ denoted by $G \times H$ where $(u,u')$ and $(v,v')$ are the order pair of the Cartesian product will be adjacent in $G \times H$ if and only if either.

- $u = v$ and $u'$ is adjacent with $v'$ in $H$, or
- $u' = v'$ and $u$ is adjacent with $v$ in $G$.

The Cartesian products of two graphs are commutative.

As time goes utilization of systems become very high, which experienced wider and complex network structure. Connection of different type of network model plays vital role in real life, so product of two existing network model gives a complex network structure with the facility of single integrated network. Such type of complex network may lead to high cost factor in communication but it experienced a high reliability also. In this paper, we mainly concentrate on
Cartesian product between $K_{n,n} \times P_r$ (where $n$ is even) and $K_{n,n} \times C_r$.

The rest of the paper organized is as follows. Section [sec:2] contains some preliminaries and definitions, Section [Sec:3] presents algorithm to label Cartesian product between $K_{n,n} \times P_r$ and $K_{n,n} \times C_r$, analysis of algorithm followed by conclusion.

II. PRELIMINARY NOTES

In this Section, we present some definitions related to graph and Cartesian product.

Definition 2.1: A simple undirected graph is defined as in which every pair of distinct vertices are connected by using a unique edge. A complete graph with $n$ vertices is denoted by $K_n$. For this example, for all vertices $x, y \in V$ there must be an edge $x, y \in E$.

Definition 2.2: A graph $G$ is called a complete bipartite graph if its vertices can be partitioned into two subsets $V_1$ and $V_2$ such that no edges have both end points in the same subset, and each vertex of $V_1$ ($V_2$) is connected with all vertices of $V_2$ ($V_1$). Here $V_1 = \{x_{11}, x_{12}, x_{13}, \ldots, x_{1m}\}$ contains $m$ vertices and $V_2 = \{y_{11}, y_{12}, y_{13}, \ldots, y_{1n}\}$ contains $n$ vertices.

A complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. We consider here $K_{n,n}$ where $n$ is even.

![Fig 2.1 Complete bipartite graph $K_{n,n}$](image)

Definition 2.3: A path is consider as a trail in which except possibly the first and last all vertices are different. Generally a trail is a walk in which all different edges are different. A walk for the length $K$ in a graph may produces a alternate sequence like $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ of vertices and edges which begins and ends with vertices. If the graph is directed, then $e_i$ is a directed arc for $v_i$ to $v_{i+1}$.

Definition 2.4: A cycle of a graph $G = (V, E)$ is denoted by $C_r$, where $V = \{v_0, v_1, \ldots, v_r\}$ be the set of vertices and $E = \{e_0, e_1, \ldots, e_r\}$ be the set of edges form a cycle if every vertex say $v_i$ is adjacent to exactly two vertices.

The Cartesian product $K_{n,n} \times P_r$ between $K_{n,n}$ and $P_r$ can be visualized in a simple way. For this product, we draw $r$ copies of $K_{n,n}$. Let $X_i = \{x_{i1}, x_{i2}, x_{i3}, \ldots, x_{in}\}$ and $Y_j = \{y_{j1}, y_{j2}, y_{j3}, \ldots, y_{jn}\}$ be the set of vertices of the $i^{th}$ copy of the graph $K_{n,n}$. The vertices of $i^{th}$ copy of $K_{n,n}$ are connected with $(i+1)^{th}$ copy of $K_{n,n}$ only as per following rule:

- $x_{i1}p_{i1}$ and $x_{i2}p_{i2}$ will be connected if $p_{i1} = p_{i2}$ and $|i_1 - i_2| = 1$.
- $y_{jiq_{i1}}$ and $y_{jij_{i2}}$ will be connected if $q_{i1} = q_{i2}$ and $|j_1 - j_2| = 1$.

Note that the set of vertices of $G = (V, E) = K_{n,n} \times P_r$ is $\bigcup_{i=1}^{n} X_i \bigcup_{j=1}^{r} Y_j$. It is clear that $x_{ip}, y_{jq} \in E$ i.e. $d(x_{ip}, y_{jq}) = 1$ for $i,j = 1, 2, 3, \ldots, n$ and $p, q = 1, 2, 3, \ldots, r$. Again $d(x_{ip}, x_{(i+1)p}) = 1$ for $i = 1, 2, 3, \ldots, r$, $p = 1, 2, 3, \ldots, n$ and $d(y_{jq}, y_{(j+1)q}) = 1$ for $j = 1, 2, 3, \ldots, r$, $q = 1, 2, 3, \ldots, n$.

For Cartesian product between $K_{n,n} \times C_r$ we have to draw the graph $K_{n,n}$, $r$ times. Here each $K_{n,n}$ has two set of vertices $X, Y$ where $|X| = n$, $|Y| = n$. Each set of vertices of $K_{n,n}$ for $r$ copies represented by $X_i = \{x_{i1}, x_{i2}, x_{i3}, \ldots, x_{in}\}$ and $Y_j = \{y_{j1}, y_{j2}, y_{j3}, \ldots, y_{jn}\}$.
Consider the graph $G = (K_{n,n} \times C_r)$, where $V = \bigcup_{i=1}^{r} V_i$ and $i = (X_i, Y_i)$, where $X_i = \{x_{i1}, x_{i2}, x_{i3}, \ldots, x_{ip}, \ldots, x_{in}\}$ and $Y_i = \{y_{i1}, y_{i2}, y_{i3}, \ldots, y_{iq}, \ldots, y_{in}\}$ for each $i = 1, 2, 3, \ldots, r$ and $p, q = 1, 2, 3, \ldots, n$. Two vertices are to be connected by the followings

- $x_{il}p_1$ and $x_{il}p_2$ will be connected if $p_1 = p_2$ and $|i - j| = 1$ or $r - 1$.
- $y_{jq_1}$ and $y_{jq_2}$ will be connected if $q_1 = q_2$ and $|j - j| = 1$ or $r - 1$.

It is clear that $x_{ip}y_{jq} \in E$ i.e. $d(x_{ip}, y_{jq}) = 1$ for $i,j = 1, 2, 3, \ldots, r$ and $p, q = 1, 2, 3, \ldots, n$. Again $d(x_{ip}, x_{(i+1)p}) = 1$ for $i = 1, 2, 3, \ldots, r$, $p = 1, 2, 3, \ldots, n$ and $d(y_{jq}, y_{(j+1)q}) = 1$ for $j = 1, 2, 3, \ldots, r$, $q = 1, 2, 3, \ldots, n$.

Fig 2.2: The graph $K_{n,n} \times P_r$

Fig 2.3: The graph $K_{n,n} \times C_r$
III. LABELING OF CARTESIAN PRODUCT BETWEEN $K_{n,n} \times P_r$ AND $K_{n,n} \times C_r$

This section contains two algorithms to cordial labeling of Cartesian product of the graph $G = K_{n,n} \times P$ and $G = (K_{n,n} \times C_r)$, followed by brief analysis with example.

1.1 Algorithm:

We consider $K_{n,n}$ as complete bipartite graph where $n \equiv 0 \mod(2)$. Assuming that each $K_{n,n}$ having two set of vertices $X_i, Y_j$ for $i, j = 1, 2, 3, ..., r$ and $|X_i| = |Y_j| = n$ where the vertices within the set are not connected but every vertices of $X_i$ are connected with every vertices of $Y_j$.

Already it is clear that $X_i$ has $n$ vertices and the vertices are $X_i = \{x_{i1}, x_{i2}, x_{i3}, ..., x_{in}\}$ similarly $Y_j = \{y_{j1}, y_{j2}, y_{j3}, ..., y_{jn}\}$. Here we propose two algorithms for cordial labeling of Cartesian product of $K_{n,n} \times P_r$ and $K_{n,n} \times C_r$ where for cycle we consider $r$ is even.

Here we propose algorithm of cordial labeling of Cartesian product between $K_{n,n} \times P_r$. Throughout the algorithm we consider $f(v) = 1$ as label of the vertex $v$. According to the drawing of Cartesian product between $K_{n,n} \times P_r$ it is clear that there will be $r$ copies of $K_{n,n}$.

Basically we have to label each copy of $K_{n,n}$ by $0$ and $1$ starting from $1^r, 2^r, 3^r, ..., r^r$ copy.

We also propose another algorithm of cordial labeling of Cartesian product between $K_{n,n} \times C_r$ considering $r$ is even. All other assumption is similar as Cartesian product between $K_{n,n} \times P_r$.

Algorithm 1. CLCBP

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<th>Algorithm for cordial labeling of $G = K_{n,n} \times P_r$</th>
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**Input:** A Cartesian product between $K_{n,n}$ and $P_r$ i.e $G = K_{n,n} \times P_r$, where $n$ is even.

**Output:** Cordial labeling of the graph $G = K_{n,n} \times P_r$.

Step 1: For $k = 1$ to $r$.

Step 2: If $k \equiv 1 \mod(2)$ then

$$f(x_{ip}) = 1 \text{ if } (i + p) \equiv 0 \mod(2)$$
$$f(x_{ip}) = 0 \text{ if } (i + p) \equiv 1 \mod(2)$$

and

$$f(y_{jq}) = 0 \text{ if } (j + q) \equiv 0 \mod(2)$$
$$f(y_{jq}) = 1 \text{ if } (j + q) \equiv 1 \mod(2)$$

Step 3: If $k \equiv 0 \mod(2)$ then

$$f(x_{ip}) = 0 \text{ if } (i + p) \equiv 1 \mod(2)$$
$$f(x_{ip}) = 1 \text{ if } (i + p) \equiv 0 \mod(2)$$

and

$$f(y_{jq}) = 1 \text{ if } (j + q) \equiv 0 \mod(2)$$
$$f(y_{jq}) = 0 \text{ if } (j + q) \equiv 1 \mod(2)$$

Step 4: End.

Algorithm 2. CLCBC

Algorithm for cordial labeling of $G = K_{n,n} \times C_r$.

**Input:** A Cartesian product between $K_{n,n}$ and $C_r$ i.e $G = K_{n,n} \times C_r$, where $n, r$ is even.

**Output:** Cordial labeling of the graph $G = K_{n,n} \times C_r$.

Step 1: For $k = 1$ to $r$.

Step 2: If $k \equiv 1 \mod(2)$ then

$$f(x_{ip}) = 1 \text{ if } (i + p) \equiv 0 \mod(2)$$
$$f(x_{ip}) = 0 \text{ if } (i + p) \equiv 1 \mod(2)$$

and

$$f(y_{jq}) = 0 \text{ if } (j + q) \equiv 0 \mod(2)$$
$$f(y_{jq}) = 1 \text{ if } (j + q) \equiv 1 \mod(2)$$

Step 3: If $k \equiv 0 \mod(2)$ then

$$f(x_{ip}) = 0 \text{ if } (i + p) \equiv 1 \mod(2)$$
$$f(x_{ip}) = 1 \text{ if } (i + p) \equiv 0 \mod(2)$$

and

$$f(y_{jq}) = 1 \text{ if } (j + q) \equiv 0 \mod(2)$$
$$f(y_{jq}) = 0 \text{ if } (j + q) \equiv 1 \mod(2)$$

Step 4: End.
1.2 Analysis of Algorithm (CLCBP & CLCBC):

In our example we consider $G = K_{4,4} \times P_4$ and already there is 4 copies of $K_{4,4}$ with the vertex set $X_i$ and $Y_j$ for $i, j = 1, 2, 3, 4$. Now just considering the $1^{st}$ and $2^{nd}$ copy where $X_1 = \{x_{11}, x_{12}, x_{13}, x_{14}\}$ and $X_2 = \{x_{21}, x_{22}, x_{23}, x_{24}\}$. As we consider the $1^{st}$ copy.

$k = 1$ and $k \equiv 1 \mod(2)$ then

$f(x_{11}) = f(x_{13}) = 1$

for $(1+1) \equiv 0 \mod(2)$ and $(1+3) \equiv 0 \mod(2)$
Now for the 2nd copy.

\[ f(x_{21}) = f(x_{23}) = 0 \text{ for } (2+1) \equiv 1 \mod(2) \text{ and } (2+3) \equiv 1 \mod(2) \]

\[ f(x_{22}) = f(x_{24}) = 1 \text{ and } (2+2) \equiv 0 \mod(2) \text{ and } (2+4) \equiv 0 \mod(2). \]

For \( Y_1 = \{y_{11}, y_{12}, y_{13}, y_{14}\} \) and \( Y_2 = \{y_{21}, y_{22}, y_{23}, y_{24}\} \).

As we consider the 1st copy.

\[ k = 1 \text{ and } k \equiv 1 \mod(2) \]

\[ f(y_{11}) = f(y_{13}) = 0 \text{ for } (1+1) \equiv 0 \mod(2) \text{ and } (1+3) \equiv 0 \mod(2) \]

\[ f(y_{12}) = f(y_{14}) = 1 \text{ and } (1+2) \equiv 1 \mod(2) \text{ and } (1+4) \equiv 1 \mod(2). \]

Now for the 2nd copy.

\[ k = 2 \text{ and } k \equiv 0 \mod(2) \]

\[ f(y_{21}) = f(y_{23}) = 0 \text{ for } (2+1) \equiv 1 \mod(2) \text{ and } (2+3) \equiv 1 \mod(2) \]

\[ f(y_{22}) = f(y_{24}) = 1 \text{ for } (2+2) \equiv 0 \mod(2) \text{ and } (2+4) \equiv 0 \mod(2). \]

Rest of the two copy will follow the same rule to label by 0 and 1. As we consider \( K_{n,n} \) for all \( n \) is even so we can observe that difference between the vertices label by 0 and the vertices label by 1 is 0. We also give the edge label matrix which shows that the difference between edge label by 0 and the edge label by 1 is 0 also. Which follow the restriction of cordial labeling.

Table 3.1: Vertex label matrix of \( K_{4,4} \times P_4 \) and \( K_{4,4} \times C_4 \):

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IV. CONCLUSION

In this paper we design an algorithm to cordial labeling of the graph obtained by Cartesian product between \( K_{4,4} \times P_4 \) and \( K_{4,4} \times C_4 \) where \( n \) is even number and we consider only cycle with even length i.e. \( r \) is even for \( C_r \). Our proposed algorithm for both the labeling successfully worked and we also analyse our algorithm with an example. For a complex type network assigning of frequency is really a tough job so we try to draw a simple way of its graphical representation. It is a very interesting problem for the researcher to label a graph in general way and considering more and more complex structure.

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